

Partial Differential Equations with Constant Coefficients

Dr. Kavita Sakure

Assistant Professor

Department of Mathematics

Govt. Digvijay Autonomous P.G. College, Rajnandgaon

Department of Mathematics

Govt. Digvijay Autonomous P.G. College, Rajnandgaon

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \quad (1)$$

in which k 's are constants, is called a *homogeneous linear partial differential equation of the n th order with constant coefficients*. It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\frac{\partial^r}{\partial x^r} = D^r$ and $\frac{\partial^r}{\partial y^r} = D'^r$.

(1) becomes $(D^n + k_1 D^{n-1} D' + \cdots + k_n D'^n)z = F(x, y)$
or briefly

$$f(D, D')z = F(x, y) \quad (2)$$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely: the *complementary function* and the *particular integral*. The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation
$$\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = 0 \quad \dots(1)$$

which in *symbolic form* is
$$(D^2 + k_1 DD' + k_2 D'^2)z = 0 \quad \dots(2)$$

Its symbolic operator equated to zero, *i.e.*,

$D^2 + k_1 DD' + k_2 D'^2 = 0$ is called the **auxiliary equation (A.E.)**

Let its root be

$$D/D' = m_1, m_2.$$

Case I. If the roots be real and distinct then (2) is equivalent to

$$(D - m_1 D')(D - m_2 D')z = 0 \quad (3)$$

It will be satisfied by the solution of

$$(D - m_2 D')z = 0, \quad \text{i.e.,} \quad p - m_2 q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \quad \text{whence} \quad y + m_2 x = a \quad \text{and} \quad z = b.$$

$\therefore$ its solution is $z = \phi(y + m_2 x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1 D')z = 0, \quad \text{i.e.,} \quad \text{by } z = f(y + m_1 x)$$

Hence the complete solution of (1) is $z = f(y + m_1 x) + \phi(y + m_2 x)$.

Case II. If the roots be equal (i.e., $m_1 = m_2$), then (2) is equivalent to

$$(D - m_1 D')^2 z = 0 \quad (4)$$

Putting $(D - m_1 D')z = u$, it becomes $(D - m_1 D')u = 0$ which gives

$$u = \phi(y + m_1 x)$$

$\therefore$ (4) takes the form $(D - m_1 D')z = \phi(y + m_1 x)$ or
 $p - m_1 q = \phi(y + m_1 x)$

This is again Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y + m_1 x)}$$

giving $y + m_1 x = a$ and $dz = \phi(a) dx$, i.e., $z = \phi(a)x + b$

Thus the complete solution of (1) is

$$z - x\phi(y + m_1 x) = f(y + m_1 x), \quad \text{i.e.,} \quad z = f(y + m_1 x) + x\phi(y + m_1 x).$$

Example Solve $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$.

Solution. Given equation in symbolic form is

$$(2D^2 + 5DD' + 2D'^2)z = 0.$$

Its *auxiliary equation* is $2m^2 + 5m + 2 = 0$, where $m = D/D'$.

which gives

$$m = -2, -1/2.$$

Here the complete solution is $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

which may be written as $z = f_1(y - 2x) + f_2(2y - x)$.

Example Solve $4r + 12s + 9t = 0$.

Solution. Given equation in symbolic form is

$$(4D^2 + 12DD' + 9D'^2)z = 0$$

for $r = \frac{\partial^2 z}{\partial x^2} = D^2 z$, $s = \frac{\partial^2 z}{\partial x \partial y} = DD' z$ and $t = \frac{\partial^2 z}{\partial y^2} = D'^2 z$.

$\therefore$ Its auxiliary equation is $4m^2 + 12m + 9 = 0$, whence

$$m = -3/2, -3/2$$

Hence the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$.

RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $(D^2 + k_1 DD' + k_2 D'^2)z = F(x, y)$ i.e.,
 $f(D, D')z = F(x, y)$.

$$\therefore \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

Case I. When

$$F(x, y) = e^{ax+by}$$

Since

$$De^{ax+by} = ae^{ax+by} ; D'e^{ax+by} = be^{ax+by}$$

$$D^2e^{ax+by} = a^2e^{ax+by} ; DD'e^{ax+by} = abe^{ax+by}$$

and

$$D'^2e^{ax+by} = b^2e^{ax+by}$$

$$\therefore (D^2 + k_1 DD' + k_2 D'^2)e^{ax+by} = (a^2 + k_1 ab + k_2 b^2) e^{ax+by}$$

Govt. Digvijay Autonomous P.G. College, Rajnandgaon

$$\text{i.e.,} \quad f(D, D')e^{ax+by} = f(a, b) e^{ax+by}$$

Operating both sides by $1/f(D, D')$, we get

$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}$$

Case II. When $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

Since $D^2 \sin(mx + ny) = -m^2 \sin(mx + ny)$

$DD' \sin(mx + ny) = -mn \sin(mx + ny)$

and $D'^2 \sin(mx + ny) = -n^2 \sin(mx + ny)$

$\therefore f(D^2, DD', D'^2) \sin(mx + ny) = f(-m^2, -mn, -n^2) \sin(mx + ny)$

Operating both sides by $1/f(D^2, DD', D'^2)$, we get

P.I. = $\frac{1}{f(D^2, DD', D'^2)} \sin(mx + ny) = \frac{1}{f(-m^2, -mn, -n^2)} \sin(mx + ny)$

Similarly about the P.I. for $\cos(mx + ny)$.

Case III. When $F(x, y) = x^m y^n$, m and n being constants.

$$\therefore \text{P.I.} = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

To evaluate it, we expand $[f(D, D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and then operate on $x^m y^n$ term by term.

Case IV. When $F(x, y)$ is any function of x and y .

$$\therefore \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

To evaluate it, we resolve $1/f(D, D')$ into partial fractions treating $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx) dx$$

where c is replaced by $y + mx$ after integration.

WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its symbolic form is $(D^n + k_1 D^{n-1} D' + \cdots + k_n D'^n)z = F(x, y)$
or briefly $f(D, D')z = F(x, y)$

Step I. To find the C.F.

(i) Write the A.E.

i.e., $m^n + k_1 m^{n-1} + \cdots + k_n = 0$ and solve it for m .

(ii) Write the C.F. as follows

Roots of A.E.	C.F.
1. $m_1, m_2, m_3 \dots$ (distinct roots)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \dots$
2. $m_1, m_1, m_3 \dots$ (two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \dots$
3. $m_1, m_1, m_1 \dots$ (three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \dots$

Step II. To find the P.I.

From the symbolic form, P.I. = $\frac{1}{f(D, D')} F(x, y)$.

(i) When $F(x, y) = e^{ax+by}$ P.I. = $\frac{1}{f(D, D')} e^{ax+by}$ [Put $D = a$ and $D' = b$]

(ii) When $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny)$$

$$D^2 = -m^2, DD' = -mn, D'^2 = -n^2]$$

(iii) When $F(x, y) = x^m y^n$, P.I. = $\frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$.

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

(iv) When $F(x, y)$ is any function of x and y P.I. = $\frac{1}{f(D, D')} F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx) dx$$

where c is replaced by $y + mx$ after integration.

Example Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$.

Solution. A.E. of the given equation is $m^2 + 4m - 5 = 0$ i.e.,
 $m = 1, -5$

$$\therefore \text{C.F.} = f_1(y + x) + f_2(y - 5x)$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 + 4DD' - 5D'^2} \sin(2x + 3y) \\
 &[\text{Put } D^2 = -2^2, DD' = -2 \times 3, D'^2 = -3^2] \\
 &= \frac{1}{-4 + 4(-6) - 5(-9)} \sin(2x + 3y) = \frac{1}{17} \sin(2x + 3y).
 \end{aligned}$$

Hence the C.S. is $z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17} \sin(2x + 3y)$.

Example Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$.

Solution. Given equation in symbolic form is

$$(D^2 - DD')z = \cos x \cos 2y.$$

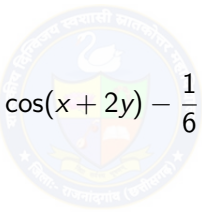
Its A.E. is $m^2 - m = 0$, whence $m = 0, 1$.

$$\therefore \text{C.F.} = f_1(y) + f_2(y + x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - DD'} \cos x \cos 2y = \frac{1}{2} \frac{1}{D^2 - DD'} [\cos(x + 2y) + \cos(x - 2y)] \\ &= \frac{1}{2} \left[\frac{1}{D^2 - DD'} \cos(x + 2y) \text{ [Put } D^2 = -1, DD' = -2] \right. \\ &\quad \left. + \frac{1}{D^2 - DD'} \cos(x - 2y) \right] \text{ [Put } D^2 = -1, DD' = 2] \\ &= \frac{1}{2} \left[\frac{1}{-1 + 2} \cos(x + 2y) + \frac{1}{-1 - 2} \cos(x - 2y) \right] \\ &= \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y) \end{aligned}$$

Hence the C.S. is

$$z = f_1(y) + f_2(y + x) + \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y).$$



Department of Mathematics
Govt. Digvijay Autonomous P.G. College, Rajnandgaon

Example Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$.

Solution. Given equation in *symbolic form* is

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

Its A.E. is $m^3 - 2m^2 = 0$, whence $m = 0, 0, 2$.

$\therefore$ C.F. = $f_1(y) + xf_2(y) + f_3(y + 2x)$

Department of Mathematics
Govt. Digvijay Autonomous P.G. College, Rajnandgaon

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^3 - 2D^2D'}(2e^{2x} + 3x^2y) = 2\frac{1}{D^3 - 2D^2D'}e^{2x} + 3\frac{1}{D^3(1 - 2D'/D)}x^2y \\
 &= 2\frac{1}{2^3 - 2.2^2(0)}e^{2x} + \frac{3}{D^3}(1 - 2D'/D)^{-1}x^2y \\
 &= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \dots\right)x^2y \\
 &= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{D}x^2.1\right) \\
 &= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{3}x^3\right) \left[\because \frac{1}{D}f(x) = \int f(x) dx\right] \\
 &= \frac{1}{4}e^{2x} + 3y\frac{x^5}{3.4.5} + 2.\frac{x^6}{4.5.6} \left[\text{since } \frac{1}{D^3}f(x) = \int \left[\int \left(\int f(x) dx\right) dx\right] dx\right]
 \end{aligned}$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Hence the C.S. is

$$z = f_1(y) + x f_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5 y + x^6).$$

Example Solve $r - 4s + 4t = e^{2x+y}$.

Solution. Given equation is $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

i.e., in symbolic form $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Its A.E. is $(m - 2)^2 = 0$, whence $m = 2, 2$.

$$\therefore \text{C.F.} = f_1(y + 2x) + x f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.
∴ to obtain the P.I., we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx) dx = \int e^{2x+(c-2x)} dx = xe^c = xe^{2x+y} [\because y = c - 2x]$$

and from $(D - 2D')z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+(c-2x)} dy = \frac{1}{2}x^2e^c = \frac{1}{2}x^2e^{2x+y}$$

Hence the C.S. is $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{1}{2}x^2e^{2x+y}$.

Example Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$.

Solution. Given equation in *symbolic form* is

$$(D^2 + DD' - 6D'^2)z = \cos(2x + y)$$

Its A.E. is $m^2 + m - 6 = 0$ whence $m = -3, 2$.

$$\therefore \text{C.F.} = f_1(y - 3x) + f_2(y + 2x).$$

Since $D^2 + DD' - 6D'^2 = -2^2 - (2 \times 1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method.

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 + DD' - 6D'^2} \cos(2x + y) = \frac{1}{(D + 3D')(D - 2D')} \cos(2x + y) \\
 &= \frac{1}{D + 3D'} \left[\int \cos(2x + c - 2x) dx \right]_{c \rightarrow y+2x} = \frac{1}{D + 3D'} \left[\int \cos c dx \right]_{c \rightarrow y+2x} \\
 &[\because y = c - mx = c - 2x] \\
 &= \frac{1}{D + 3D'} \times x \cos(y + 2x) = \left[\int x \cos(c + 3x + 2x) dx \right]_{c \rightarrow y-3x} \\
 &= \left[\int x \cos(5x + c) dx \right]_{c \rightarrow y-3x} \\
 &= \left[\frac{x \sin(5x + c)}{5} + \frac{\cos(5x + c)}{25} \right]_{c \rightarrow y-3x} \quad [\text{Integrating by parts}] \\
 &= \frac{x}{5} \sin(5x + y - 3x) + \frac{1}{25} \cos(5x + y - 3x) = \frac{x}{5} \sin(2x + y) \\
 &+ \frac{1}{25} \cos(2x + y)
 \end{aligned}$$

Hence the C.S. is

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5} \sin(2x + y) + \frac{1}{25} \cos(2x + y).$$

Department of Mathematics
Govt. Digvijay Autonomous P.G. College, Rajnandgaon